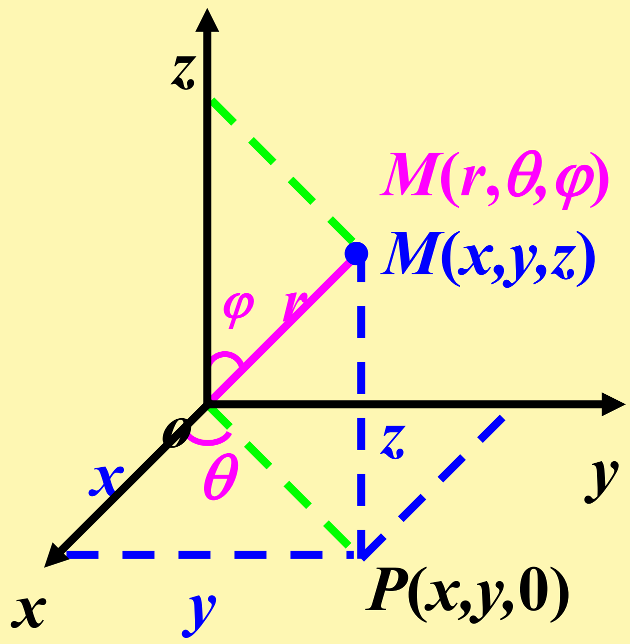
第八章　重积分

8.3　重积分的计算

8.3.4　球面坐标系下的三重积分的计算法

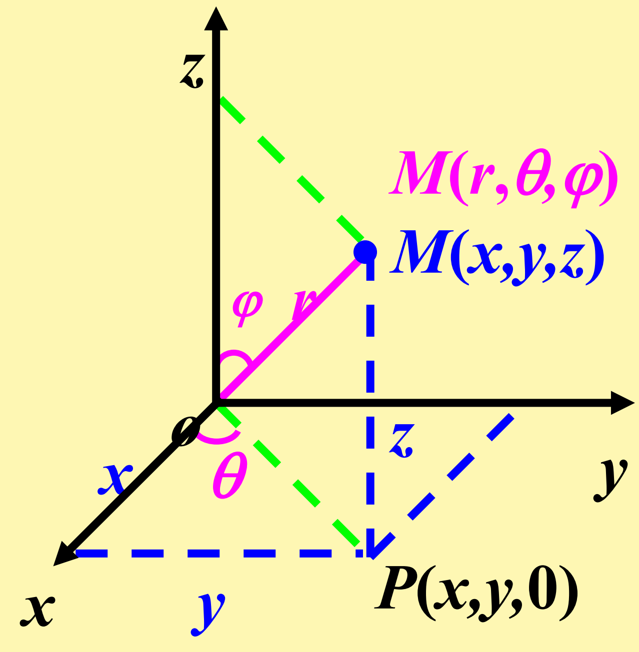
一、球面坐标

设$M(x,y,z)$为空间内一点,则点M也可用这样三个有次序的数r,φ,θ来确定。



r为原点到M间的距离。

φ为有向线段$\overrightarrow{OM}$与z轴正向所夹的角。

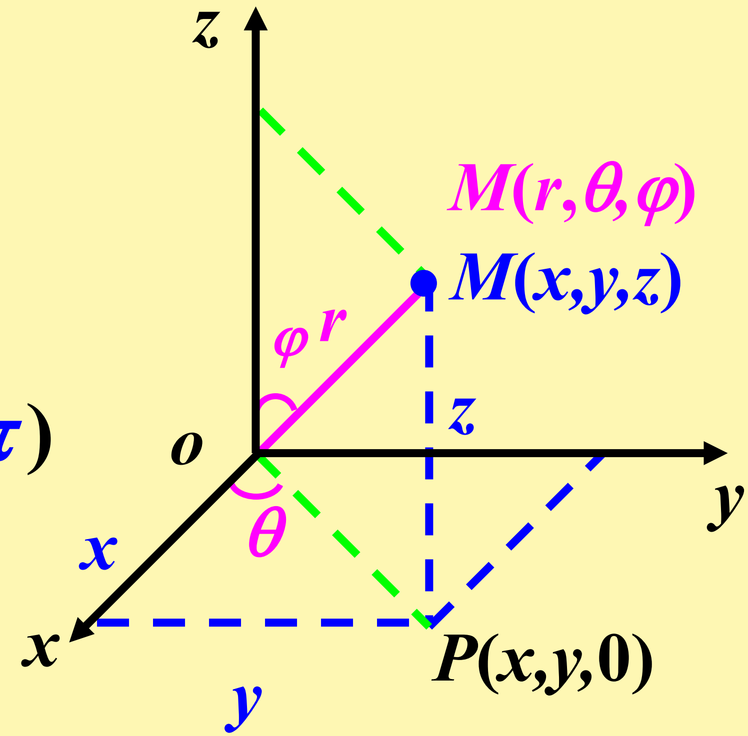


θ为从正z轴来看自x轴按逆时针方向转到有向线段$\overrightarrow{OP}$,这里P是点M在xoy平面上的投影点。

这样三个数r,φ,θ叫做点M的球面坐标。

①球面坐标的变化范围

$$\begin{cases} 0 \le r < +\infty, \\ 0 \le \varphi \le \pi, \\ 0 \le \theta \le 2\pi \quad (-\pi \le \theta \le \pi) \end{cases}$$



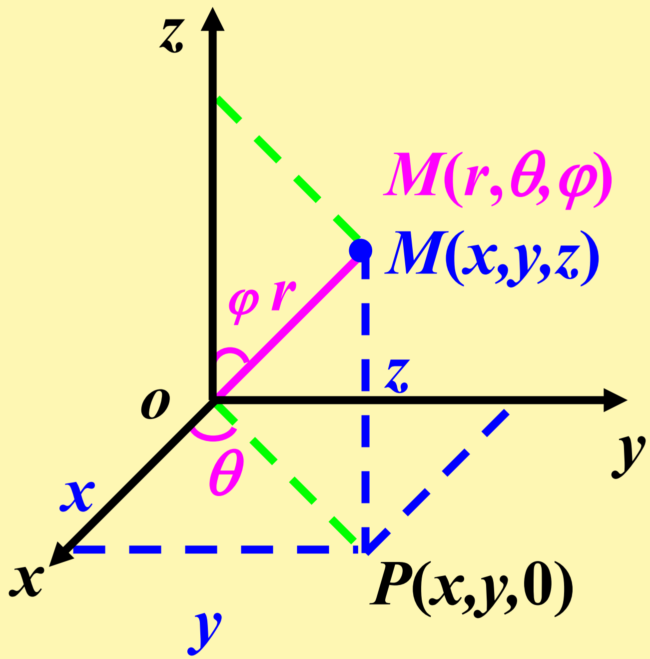
②三组坐标面

$r=$常数，即以原点为球心的球面。

$\varphi=$常数，即以原点为顶点、z轴为轴的圆锥面。

$\theta=$常数，即边为z轴的半平面。

③点M的直角坐标与球面坐标的关系为



$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$x^2 + y^2 = r^2\sin^2\varphi$$

$$x^2 + y^2 + z^2 = r^2$$

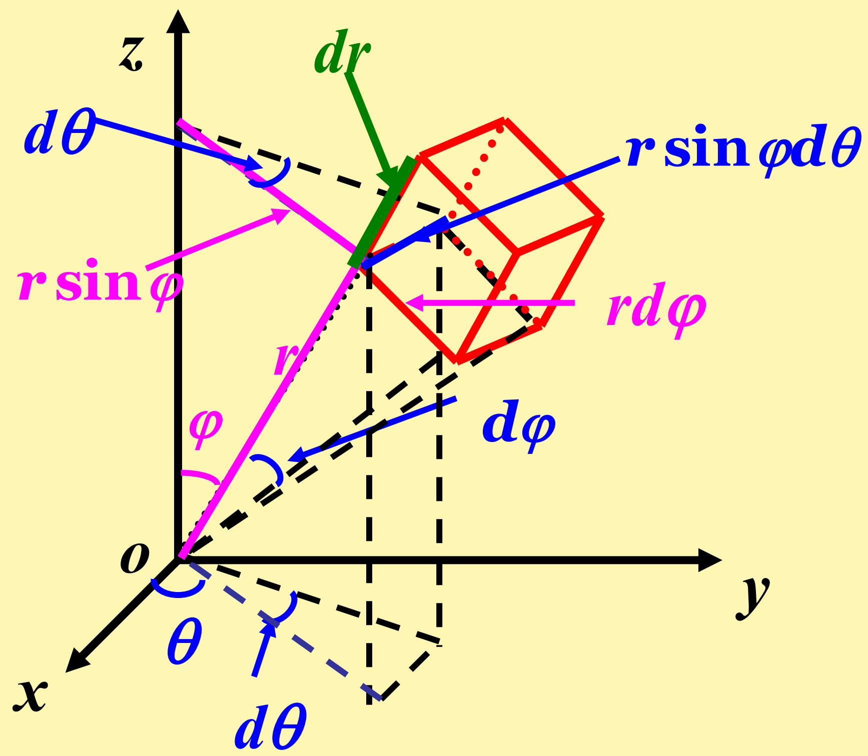
④球面坐标下的体积元素 $dv = r^2\sin\varphi\, dr\, d\varphi\, d\theta$

为了把三重积分中的变量从直角坐标变换为球面坐标，用三组坐标平面

$r=$常数，$\varphi=$常数，

$\theta=$常数

把积分区域Ω分成许多小闭区域。

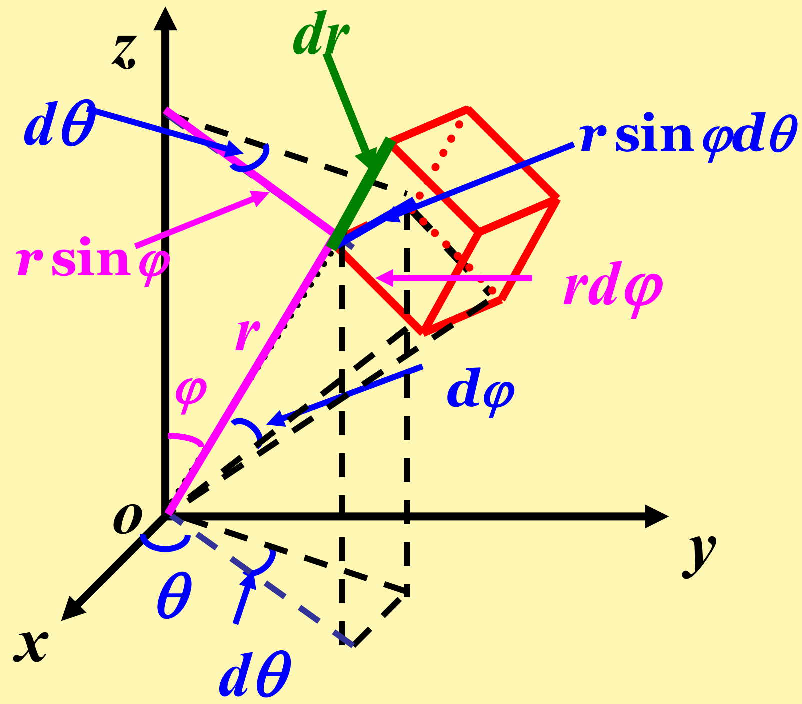


考虑由r,φ,θ各取得微小增量$dr,d\varphi,d\theta$所成的六面体的体积(如图)。不计高阶无穷小，可把这个六面体看作长方体。

经线方向的长为 $rd\varphi$,

纬线方向的宽为 $r\sin\varphi d\theta$,

向径方向的高为 dr。



于是，小六面体的体积为

$$dv = r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

这就是球面坐标系中的体积元素。

二、 三重积分的球面坐标形式

坐标变换：$x = r\sin\varphi\cos\theta,\ \ y = r\sin\varphi\sin\theta,$

$$z = r\cos\varphi$$

$$\iiint_{\Omega} f(x,y,z)dxdydz = \iiint_{\Omega} F(r,\varphi,\theta)\left|J\right|drd\varphi d\theta$$

其中$F(r,\varphi,\theta) = f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)$

$$J = \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = r^2\sin\varphi$$

$$\iiint\limits_{\Omega} f(x,y,z)dxdydz = \iiint\limits_{\Omega} F(r,\varphi,\theta)r^2\sin\varphi drd\varphi d\theta$$

一般是化为先r，再φ，最后θ的 三次积分

当原点在 Ω 内时,有

$$0 \le r \le r(\varphi,\theta), 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi,$$

$$\iiint\limits_{\Omega} f(x,y,z)dV = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{r(\varphi,\theta)} F(r,\varphi,\theta)r^2\sin\varphi dr$$

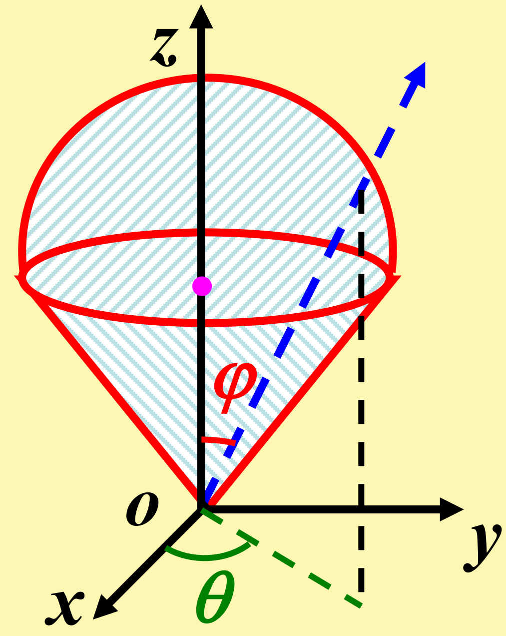
例如，半径为R的球体的体积

$$V = \iiint\limits_{\Omega} dV = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{R} r^2\sin\varphi dr$$

$$= 2\pi \cdot 2 \cdot \frac{R^3}{3} = \frac{4}{3}\pi R^3$$

例1 将$\iiint\limits_{\Omega} f(x,y,z)dv$化为球面坐标系下的三次积分形式,其中$\Omega$为:

(1) $\Omega:\begin{cases} z=\sqrt{R^2-x^2-y^2} \\ z=\sqrt{x^2+y^2} \end{cases}$ 所围区域



$$\Omega:\begin{cases} 0\le r\le R \\ 0\le \varphi\le \dfrac{\pi}{4} \\ 0\le \theta\le 2\pi \end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)dV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{R}F(r,\varphi,\theta)r^2\sin\varphi dr$$

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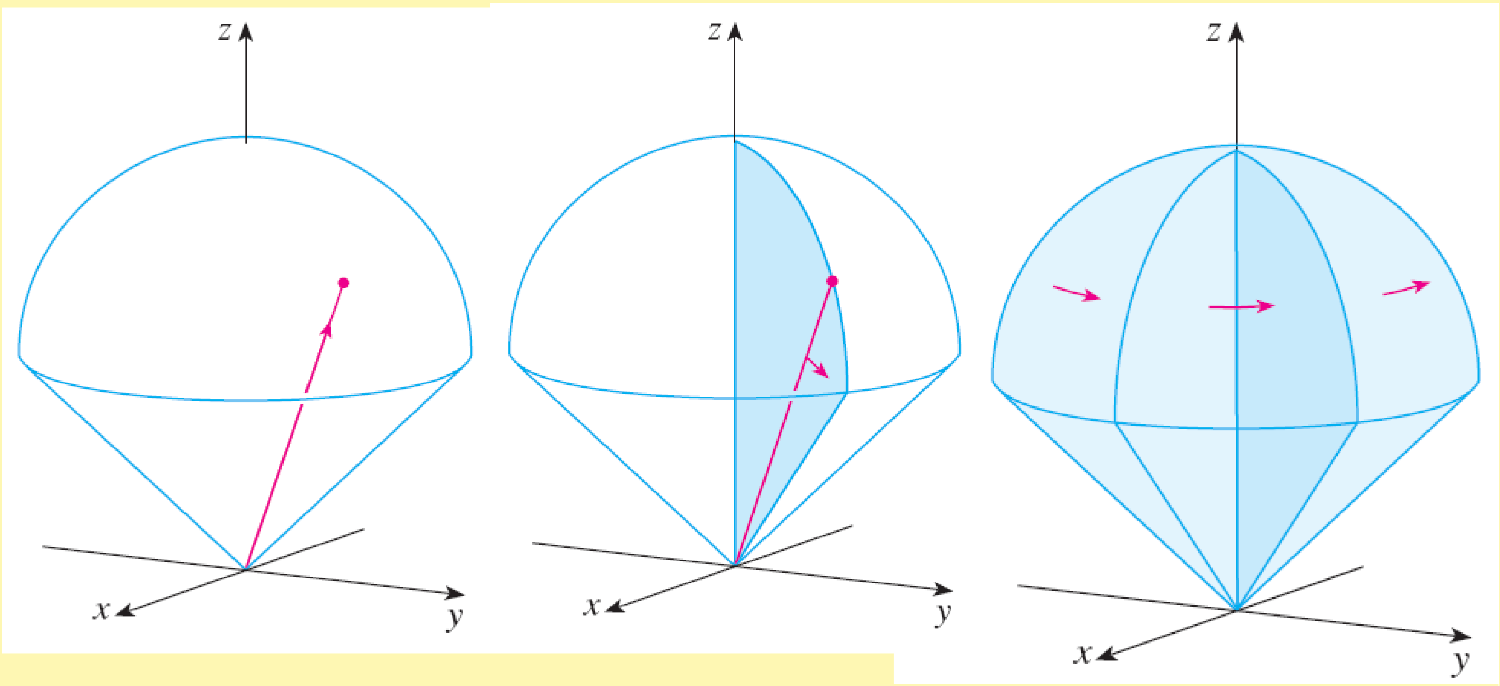
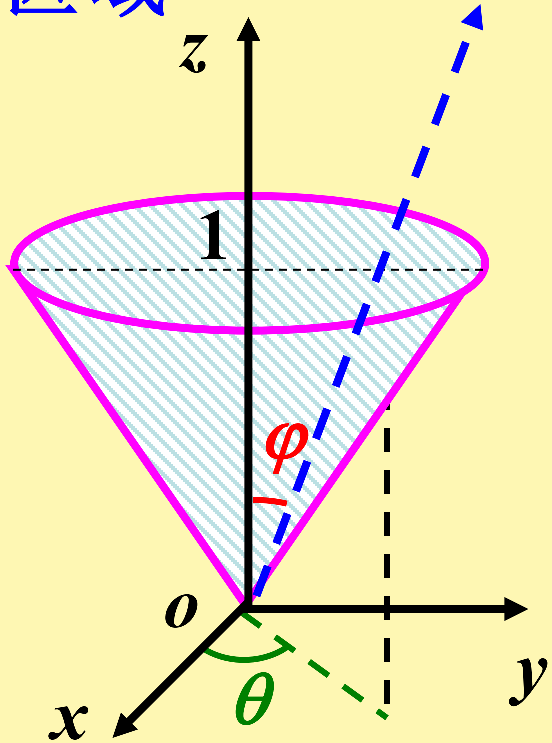


(2) $\Omega:\begin{cases} z = k\sqrt{x^2+y^2}\,(k>0) \\ z = 1 \end{cases}$ 所围区域



$\because\ k\sqrt{x^2+y^2} \le z \le 1$

$\therefore \Omega: 0 \le r \le \dfrac{1}{\cos\varphi}$,

$0 \le \varphi \le \arctan\dfrac{1}{k},\ \ 0 \le \theta \le 2\pi$

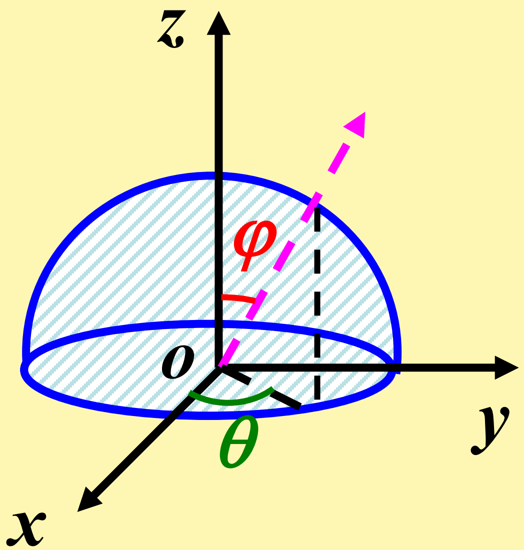
$$\therefore \iiint\limits_{\Omega} f(x,y,z)dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\arctan\frac{1}{k}} d\varphi \int_0^{\frac{1}{\cos\varphi}} F(r,\varphi,\theta) r^2 \sin\varphi\, dr$$

例2 先将积分化为球面坐标的累次积分,再求其积分值。

$$(1) I=\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}dy\int_{0}^{\sqrt{R^2-x^2-y^2}}(x^2+y^2)dz$$

解 (1) Ω 是以原点为球心,以R为半径的上半球面与xoy面所围成的空间区域。



$$\Omega: 0\le r\le R, 0\le\varphi\le\frac{\pi}{2}, 0\le\theta\le 2\pi$$

$$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^R r^2\sin^2\varphi\cdot r^2\sin\varphi dr$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^3\varphi d\varphi\int_0^R r^4 dr=\frac{4}{15}\pi R^5$$

(2) $\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\,dv,\ \Omega: x^2+y^2+z^2 \le z$

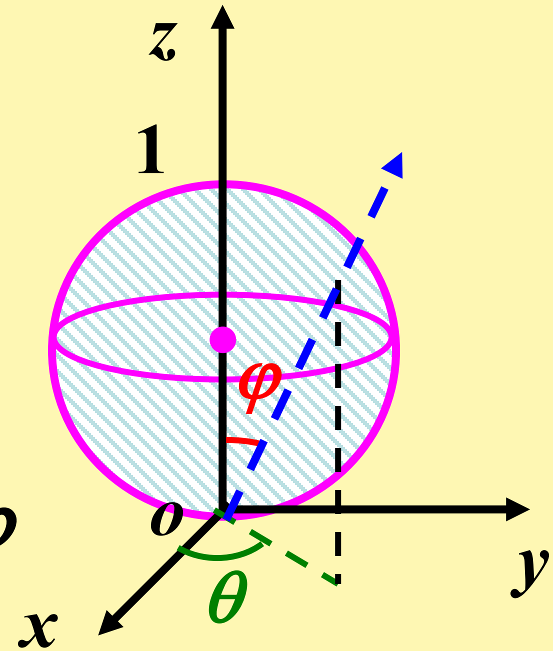
解 Ω: $0 \le r \le \cos\varphi,\quad 0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi$

$$\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\,dv$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} r \cdot r^2 \sin\varphi\, dr$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} [\sin\varphi \cdot \int_0^{\cos\varphi} r^3 dr] d\varphi$$

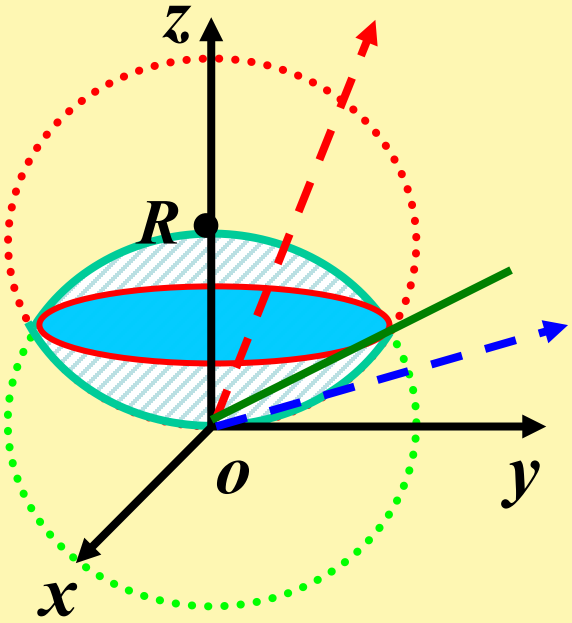
$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\varphi \frac{\cos^4\varphi}{4} d\varphi = \frac{\pi}{10}$$



例3 求三重积分 $\iiint\limits_{\Omega} z^2 dv$

Ω：由 $x^2+y^2+z^2 \le R^2$ 与

$x^2+y^2+z^2 \le 2Rz$ 所围



例3 求三重积分$\iiint\limits_{\Omega} z^2 dv$

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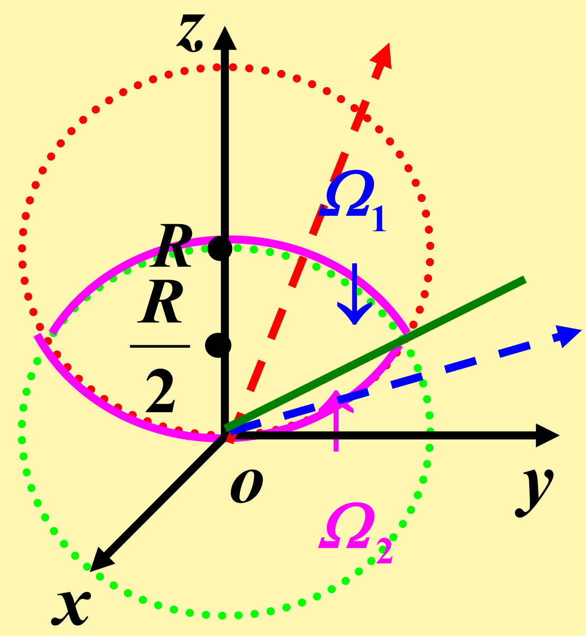


解 先求两曲面的交线方程：

$$\begin{cases} x^2+y^2=R^2-z^2 \\ x^2+y^2=2Rz-z^2 \end{cases} \Rightarrow \begin{cases} z=\dfrac{R}{2} \\ x^2+y^2=\dfrac{3}{4}R^2 \end{cases}$$

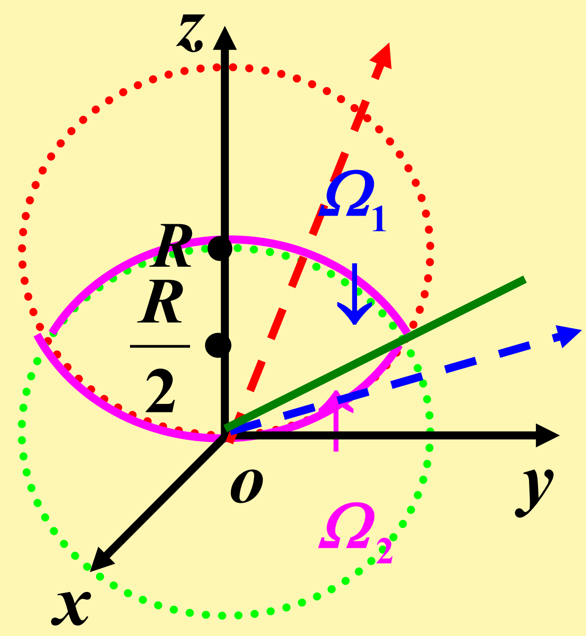
$$(0,\ \frac{\sqrt{3}}{2}R,\ \frac{R}{2})\quad \Omega_1: 0\le r\le R,\ 0\le\varphi\le\frac{\pi}{3},\ 0\le\theta\le 2\pi$$

$$\Omega_2: 0\le r\le 2R\cos\varphi,\ \frac{\pi}{3}\le\varphi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi$$

$\Omega_1: 0 \le r \le R,\ \ 0 \le \varphi \le \dfrac{\pi}{3}$

$\Omega_2: 0 \le r \le 2R\cos\varphi, \dfrac{\pi}{3} \le \varphi \le \dfrac{\pi}{2}$

$0 \le \theta \le 2\pi$



解法一 用球面坐标

特殊方法确定$\dfrac{\pi}{3}$

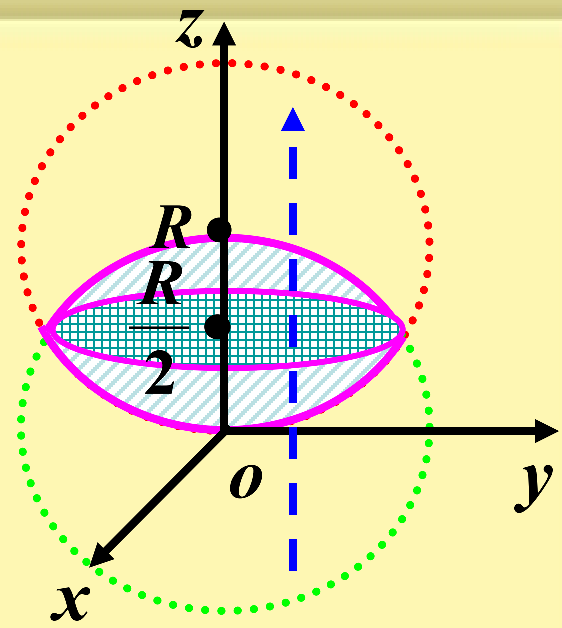
$$\iiint_{\Omega} z^2 dv = \iiint_{\Omega_1} z^2 dv + \iiint_{\Omega_2} z^2 dv$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} d\varphi \int_0^{R} r^2\cos^2\varphi \cdot r^2 \sin\varphi\, dr$$

$$+ \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2R\cos\varphi} r^2\cos^2\varphi \cdot r^2 \sin\varphi\, dr = \frac{59\pi}{480} R^5$$

$\Omega: x^2+y^2+z^2 \le R^2$

与 $x^2+y^2+z^2 \le 2Rz$ 所围



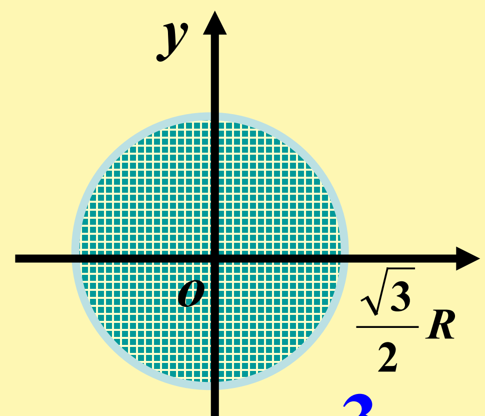
解法二　用柱面坐标系

$$\iiint_{\Omega} z^2 dv = \int_0^{2\pi} d\theta \int_0^{\frac{\sqrt{3}R}{2}} r dr \int_{R-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} z^2 dz$$

$$= 2\pi \int_0^{\frac{\sqrt{3}R}{2}} r \cdot \frac{1}{3}[(\sqrt{R^2-r^2})^3 - (R-\sqrt{R^2-r^2})^3]dr$$

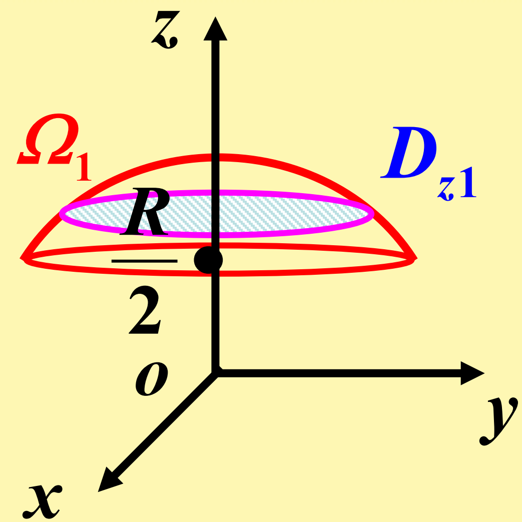
(令 $r = R\sin t, dr = R\cos t dt$)

$$= \frac{59\pi}{480} R^5$$



$D_{xy}: x^2+y^2 \le \frac{3}{4}R^2$

解法三 用切片法 (先重后单)



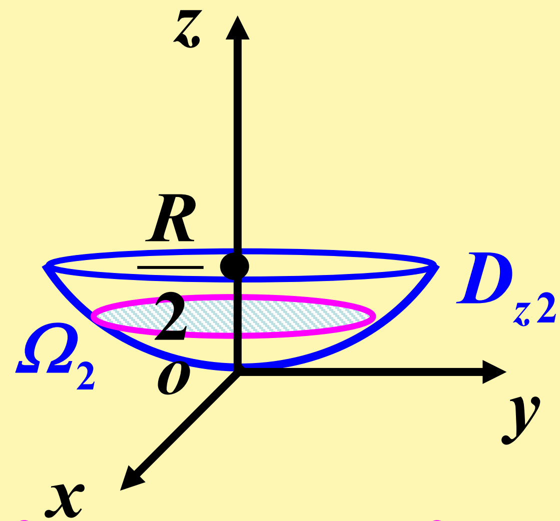
$$\iiint\limits_{\Omega} z^2 dv = \iiint\limits_{\Omega_1} z^2 dv + \iiint\limits_{\Omega_2} z^2 dv$$

$$= \int_{\frac{R}{2}}^{R} z^2 dz \iint\limits_{D_{z1}} dxdy + \int_{0}^{\frac{R}{2}} z^2 dz \iint\limits_{D_{z2}} dxdy$$

$$D_{z1}: x^2 + y^2 \le (R^2 - z^2)$$

$$= \int_{\frac{R}{2}}^{R} z^2 \pi (R^2 - z^2) dz$$

$$+ \int_{0}^{\frac{R}{2}} z^2 \pi (2Rz - z^2) dz$$



$$= \frac{59\pi}{480} R^5 。$$

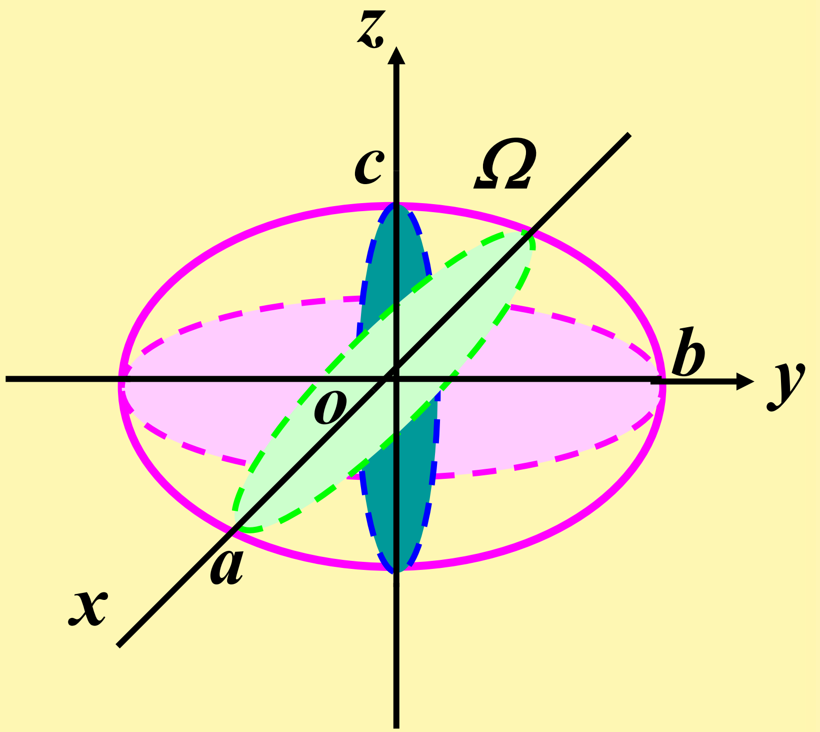
$$D_{z2}: x^2 + y^2 \le (2Rz - z^2)$$

例4 计算积分 $\iiint\limits_{\Omega} dv$，其中$\Omega$：$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\le 1$

椭球体体积

解 利用广义球面坐标系

$$\begin{cases} x = ar\cos\theta\sin\varphi \\ y = br\sin\theta\sin\varphi \\ z = cr\cos\varphi \end{cases}$$



$$J=\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=abcr^2\sin\varphi$$

$$\iiint\limits_{\Omega} dv=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1 abcr^2\sin\varphi dr=\frac{4\pi abc}{3}$$

体积元素 $dv = abcr^2\sin\varphi drd\varphi d\theta$

小结三重积分的计算方法：

基本方法：化三重积分为三次积分计算。

关键步骤：

(1)坐标系的选取

(2)积分顺序的选定（直角）

(3)定出积分限

坐标系	适用范围	体积元素	变量代换
直角坐标	长方体 四面体 任意形体	$dxdydz$	$x = x$ $y = y$ $z = z$
柱面坐标	柱形体域 锥形体域 抛物体域	$rdzdrd\theta$	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$
球面坐标	球形体域 或其中一 部分	$r^2\sin\varphi drd\varphi d\theta$	$x = r\cos\theta\sin\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\varphi$